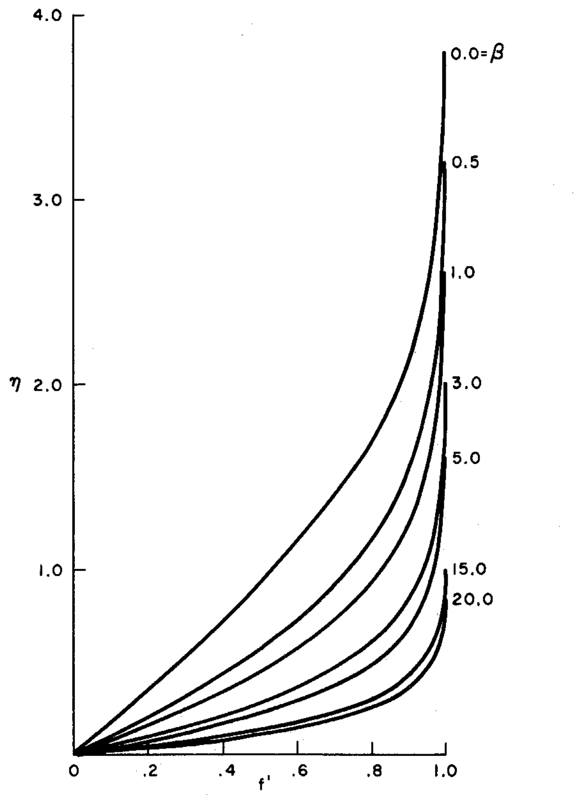

Fig. 2 Effect of accelation on velocity profiles. $Pr = 0.715$, $E = 0.9$, $\omega = 0.5$.

is readily estimated if it is noted that

$$\phi^{-1} \leq \exp(-Pr\eta^2/2C_0) \quad (6)$$

with the equality holding only at the inner boundary, $\eta = 0$ Thus

$$\int_0^\infty (C\phi)^{-1} \int_0^\eta (Cf'f'')\phi d\bar{\eta} d\eta > \int_0^\infty (C\phi)^{-1} \times \int_0^\eta (Cf'f'')' d\bar{\eta} d\eta = \int_0^\infty f'f''\phi^{-1} d\eta > \int_0^\infty [f'f'' \exp(-Pr\eta^2/2C_0)] d\eta = \tfrac{1}{2}C_0 \int_0^\infty f'^2 Pr\eta \exp(-Pr\eta^2/2C_0) d\eta > 0$$

Since $Pr < 1$ and $E > 0$ we have from Eq. (4),

$$1 - g_0 > 0 \quad (7)$$

Also, direct integration of the right hand side of Eq. (4) yields

$$1 - g_0 = (1 - Pr)E - 2(1 - Pr)PrE \times \int_0^\infty (C\phi)^{-1} \int_0^\eta f''f'f\phi d\bar{\eta} d\eta \quad (8)$$

In the inner integral all the functions with the exception of f'' are non-negative. In situations where there is a velocity overshoot f'' has to be negative locally, however the integral of f'' must still be positive. Thus the second term on the right hand side of Eq. (8) is positive and

$$1 - g_0 < (1 - Pr)E \quad (9)$$

Recalling now that the recovery factor is defined by the relation $(1 - g_0) = (1 - r)E$, we obtain from Eqs. (7) and (9)

$$Pr < r < 1.0 \quad (10)$$

Thus the recovery factor is less than unity, which it approaches from below as $Pr \rightarrow 1.0$. As $\beta \rightarrow \infty$ a singular perturbation problem obtains in which the velocity boundary-layer thickness is of the order of $\beta^{-1/2}$ with respect to a total enthalpy layer of order unity. Figure 2 shows the steepening velocity profiles as β increases. Now $ff''\phi \rightarrow 0$ as $\eta \rightarrow \infty$ so that the integrand of the inner integral in Eq. (8) is finite and tends to zero with increasing η. Also the measure of the interval of integration approaches zero as $\beta \rightarrow \infty$. Therefore the inner integral tends to zero as $\beta \rightarrow \infty$, and

$$r \rightarrow Pr \quad \text{as} \quad \beta \rightarrow \infty \quad (11)$$

We have shown analytically that for all finite β the recovery factor is less than unity and that it tends to the Prandtl number as $\beta \rightarrow \infty$. Our numerical data confirms these conclusions. The conjectures of Dewey and Gross[1] and Back[2] are therefore incorrect.

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Decay of a Boundary-Layer Induced Shock

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Introduction

THE large Reynolds number flow over a flat plate is conceptually well established. The outer, inviscid flow, behaves as though it were displaced by a body equivalent to the displacement thickness of the viscous boundary layer. This effective body generates a shock wave in the inviscid flow and the shock is attenuated by expansion waves which ultimately reduce its strength to zero. A solution for the asymptotic decay of the shock has application in the gas dynamic laser[1] (GDL) where gas density inhomogeneities may cause distortion of the laser beam.

Considering only boundary layers whose displacement thickness (δ^*) increases algebraically with distance (x)

$$\delta^* = Ax^N \quad (0 \leq N \leq 1) \quad (1)$$
$$(A = \text{constant})$$

the strong shock solution may be generated using the hypersonic similarity solutions of Sedov.[2] This, of course, requires a priori knowledge of the boundary-layer growth which is

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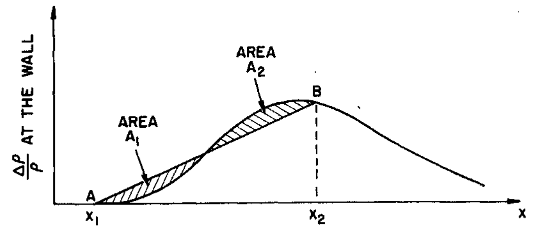

Fig. 1 Whitham's area rule (two-dimensional).

coupled to the shock strength in this regime[8] (strong interaction regime).

When the disturbance generated by the boundary layer decays to that describable by acoustics (weak interaction regime), the boundary layer displacement thickness is calculated to lowest order by neglecting the presence of the shock and the first order shock strength is determined from the lowest order boundary layer solution. Application of linear theory to power law bodies yields the density disturbance at the wall

$$\left.\frac{\Delta\rho}{\rho}\right|_w = \frac{M_\infty{}^2 N A x_w{}^{N-1}}{(M_\infty{}^2 - 1)^{1/2}} \tag{2}$$

where x_w indicates the value of x at the wall. Below, emphasis is placed on the shock at distances far from the boundary layer and a uniformly valid acoustic solution is obtained for the shock strength.

Method of Solution

As first formalized by Lighthill,[4] acoustic solutions can be made uniformly valid by writing the acoustic solution in characteristic coordinates and determining the location of the characteristic to first order. Shocks are formed by the coalescence of waves and the conservation laws require that the shock bisect the characteristic directions. Friedrichs[5] and Lighthill[6] have used acoustics to determine the decay of the shock wave generated by a two-dimensional wing in an otherwise uniform flow. The initial shock is attenuated by expansion waves originating on the wing. The results, valid far from the body, predict that the shock decays as $cx^{-1/2}$ where x is the coordinate parallel to the flow and c is a constant that is dependent on the wing size and shape. Whitham[7] extended these methods to include finite axisymmetric bodies, finding the shock to decay as $x^{-3/4}$.

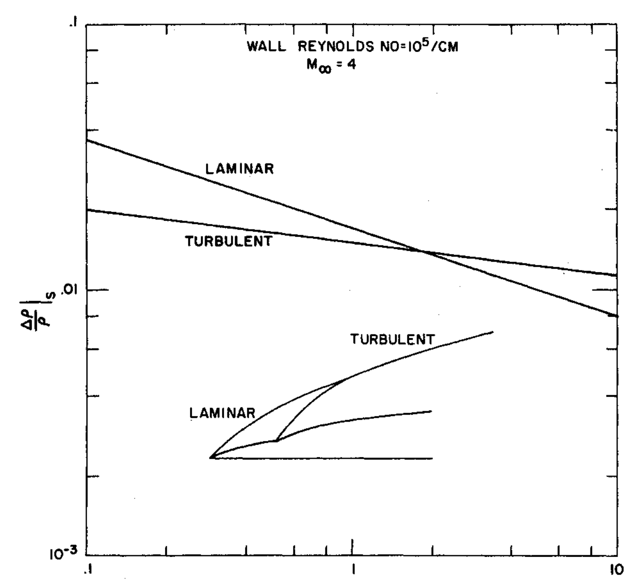

Fig. 2 Density perturbation behind a boundary-layer driven shock.

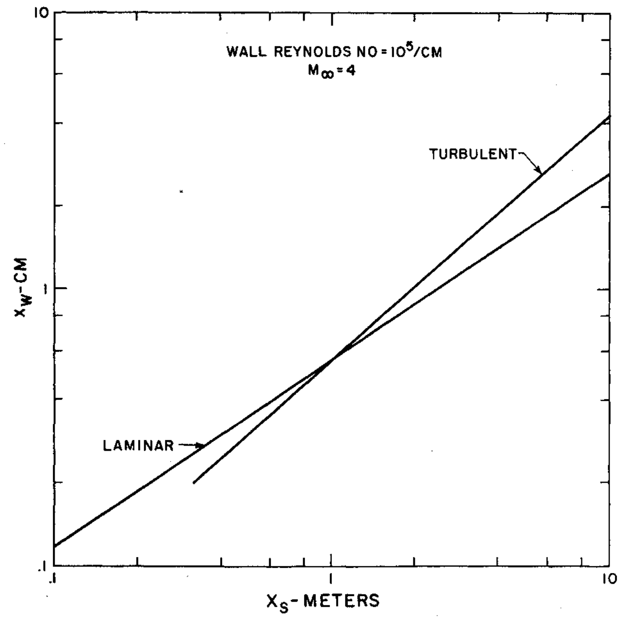

Fig. 3 Origin of the waves striking the shock at x_s.

In considering the shock generated by an infinite body, we note that it is the waves at infinity that attenuate the shock at infinity. Thus, it is not an initial value problem but one of asymptotics. This leads to different decay laws. However, the general approach of Friedrichs, Whitham and Lighthill may still be used to determine the asymptotic decay of a shock generated by a two-dimensional power law body.

Whitham[7] has derived a general result for the decay of two-dimensional and axisymmetric disturbances which is often referred to as the "area rule." Its application to the two-dimensional case is illustrated in Fig. 1. When the waves originating at x_1 and x_2 intersect, a shock is required. The motion of the waves and shock are compatible with the equations of motion when area $A_1 = A_2$. The x location of the shock is

$$x_s = \frac{2(M_\infty{}^2 - 1)}{(\gamma + 1)M_\infty{}^2\theta_{AB}} \tag{3}$$

where θ_{AB} is the slope of the line AB in Fig. 1.

The "area rule" may be applied to that disturbance represented by Eq. (2). The singularity at $x = 0$ does imply a failure of linear theory. However, the singularity is integrable and asymptotically, the area in the region of the singularity represents an ever decreasing fraction of the total area. The author[8] has generated the asymptotic solution to the differential equations and obtained results identical to those given below.

Application of Whitham's rule to the disturbances created by a power law body yields $\theta_{AB}(\Delta\rho/\rho)$.

$$\theta_{AB} = \frac{1}{2}\left[\frac{M_\infty{}^2 N^N A}{(M_\infty{}^2 - 1)^{1/2}}\left(\frac{\Delta\rho}{\rho}\right)^{N-2}\right]^{1/(N-1)}$$

Since $\Delta\rho/\rho$ is constant on the wave, $\Delta\rho/\rho$ behind the shock follows from Eq. (3).

$$\left.\frac{\Delta\rho}{\rho}\right|_s = \left[\frac{M_\infty{}^{2N} A N^N}{(M_\infty{}^2 - 1)^{N-1/2}}\left(\frac{\gamma+1}{4}\right)^{N-1}\right]^{1/(2-N)} \times \frac{1}{x_s{}^{(1-N)/(2-N)}} \tag{4}$$

In the limit of $N \to 0$, all of the characteristics must come from a disturbance at $x_w = 0$. Such a disturbance must create a shock as described by the sonic boom theory. Examining these results as $N \to 0$, it is seen that the $x^{-1/2}$ decay of the sonic boom is recovered. However, for $N > 0$, the disturbance decays more slowly than $x^{-1/2}$. This is a consequence of the ever increasing displacement thickness. Since the boundary layer does not allow the flow to expand to its original state, the expansion from the viscous layer is slower than a centered expansion and cannot decrease the shock strength as rapidly.

Results

These results may now be applied to that disturbance generated by a viscous boundary layer. The displacement thickness in a compressible fluid is a function of several parameters. Since we only wish to illustrate the theory, the displacement thickness for an incompressible flow is used.

For an external flow with a Reynolds number of 10^5/cm, the displacement thickness corresponding to an incompressible laminar boundary layer is

$$\delta^* = 6 \times 10^{-3} x^{1/2} \qquad (x - cm)$$

Similarly, the displacement thickness for a turbulent boundary layer is:

$$\delta^* = 4 \times 10^{-3}(x)^{0.86}$$

The density perturbations behind a shock attenuated by laminar and turbulent boundary layers in a Mach 4 flow are illustrated in Fig. 2. A schematic diagram of the shock waves is also presented. The laminar results are applicable beyond the boundary-layer transition length because the waves from the boundary layer require a finite time to reach the shock. The origin of the waves striking the shock at x_s is indicated in Fig. 3.† Transition to turbulence at a Reynolds number of 10^5 would correspond to $x_w = 1$ cm. Figure 3 indicates that this would be felt by the shock about 2 m downstream. Thus, the laminar results in Fig. 2 are appropriate for $x_s < 2$ m. Laminar results are valid beyond $x_s = 2$ m when transition occurs at higher Reynolds numbers.

The analysis of the shock that is driven by the turbulent boundary layer does not include the effect of the initial "laminar" shock. Thus, it is valid only as $x_s \to \infty$ and it is not an accurate description of the flow just downstream of the point where the two shocks coalesce.

Conclusions

Whitham's "area rule" has been applied to disturbances generated by power law bodies in an otherwise uniform two-dimensional flow. It has been shown that the shocks produced by such bodies decay more slowly than those from bodies of finite extent.

The results have been applied to disturbances generated by a viscous boundary layer. For the case cited previously, the application of acoustics is valid for $x_s > 1$ cm $(\Delta\rho/\rho|_s < 0.1)$ and density disturbances of the order of a few percent are shown to exist for several meters.

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† Since $\Delta\rho/\rho$ is constant on a wave, $x_w(x_s)$ is determined from Eqs. (2) and (4).

Slipstream Formed by a Supersonic Source in a Hypersonic Stream

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Many fluid flow phenomena exhibit somewhat similar flowfield features. For example, the flow pattern produced by a high-pressure jet issuing into an oncoming stream is somewhat similar to that of a retrofiring rocket nozzle descending into a thin atmosphere at hypersonic speeds, or to a high intensity explosion moving at great speeds in a fluid ambient. Besides its inherent academic interest, some general features of the aforementioned flows can be assessed by considering a simplified flow model which approximates these flows in a certain limit.

To this end we consider the flowfield produced by a supersonic source of radius r^* which spews out fluid into an oncoming hypersonic stream as shown in the lower half of Fig. 1. In the hypersonic limit ($M_\infty \to \infty$, $\gamma \to 1$) and when the source strength becomes large, so that the ratio of the centerline distance of the shock system from the center of the source to the source radius is large ($r_0/r^* \gg 1$), the shock surfaces and contact surface can be considered as a single surface. This contact surface separates a uniform freestream from an equally isentropic source flow. It is the purpose of this Note to determine the shape of this limiting surface.

The shape of the slipstream is uniquely determined from the requirement that the pressure be equal on both sides of the slip-surface and from geometric constraints. The geometric constraint arise from the fact that the slip-surface must lie between the vertical and the normal to the radial vector as shown in the upper half of Fig. 1.

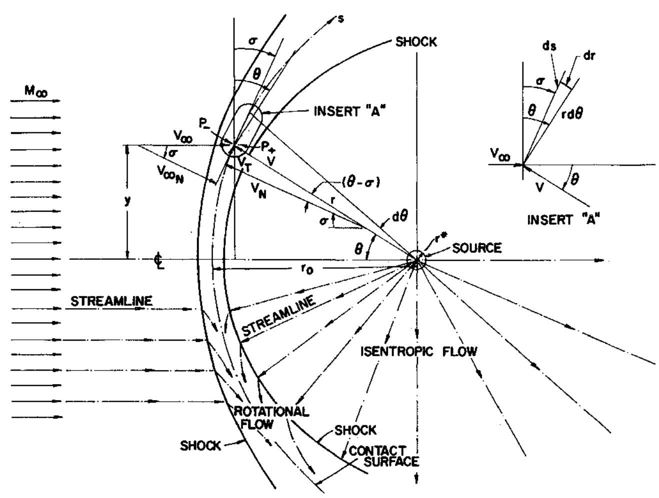

Fig. 1 General features of the flowfield produced by a supersonic source in a hypersonic stream.

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